ETMAG

CORONALECTURE 11

Determinant of a Matrix

Inverse Matrix

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Definition.
Determinant (*det* for short) is a function defined on the set of all square matrices (i.e. $n \times n$ matrices, n=1,2, …) over a field $\mathbb{K}$ into $\mathbb{K}$. The definition is inductive with respect to n:

1. if n=1, $A=[a_{1,1}]$ then $\det(A) = a_{1,1}$
2. if n>1

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(A_{i,1})$$

where $A_{i,j}$ denotes the matrix obtained from A by the removal of row number i and column j. det(A) is also denote by |A|.

The formula is known as *Laplace expansion on column 1*.

Notice that in the above definition we only use the symbol $A_{i,j}$ in the case j=1.

The size of $A_{i,j}$ is $(n\text{-}1)\times(n\text{-}1)$.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$.

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+1} a_{i,1} \det\left(A_{i,1}\right) = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

In particular, $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 2 \cdot (-2) - 1 \cdot 3 = -7$

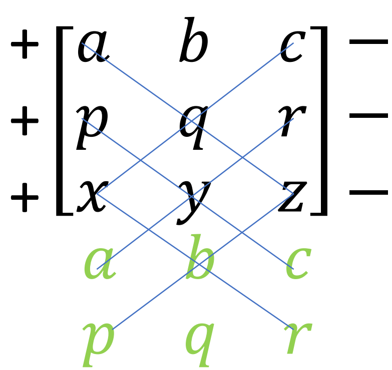
Example.

2. $det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = (-1)^{1+1} a\, det\begin{bmatrix} q & r \\ y & z \end{bmatrix} + (-1)^{2+1} p\, det\begin{bmatrix} b & c \\ y & z \end{bmatrix}$

$+ (-1)^{3+1} z\, det\begin{bmatrix} b & c \\ q & r \end{bmatrix} = a(qz - ry) - p(bz - cy) + x(br - qc) =$

$aqz + pyc + xbr - cqx - rya - zbp$. The last formula is known as the *Sarrus Rule.*

BEWARE !. It only works for 3×3 matrices.



Theorem.

For every $j = 1,2, \dots, n$ and for every $n \times n$ matrix A

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j})$$

Proof. Omitted.

Remark. The theorem says that instead of Laplace expansion on column 1 we can do Laplace expansion on column *j* - for any *j* within reason.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ by *Laplace expansion* on column 2.

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+2} a_{i,2} \det\left(A_{i,2}\right) = -a_{1,2}a_{2,1} + a_{2,2}a_{1,1}$$

Theorem. (determinant versus transposition)

For every matrix A $\det(A) = \det(A^T)$

Proof. Omitted.

Remark. The theorem says (indirectly) that instead of Laplace expansion on columns we can do Laplace expansion on rows.

Example.

1. Find $det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ by *Laplace expansion* on row 1.

$$\det(A) = \sum_{i=1}^{2} (-1)^{1+i} a_{1,i} \det(A_{1,i}) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

Theorem. (*det* versus EROS)

For every matrix A

1. If A ~ ($r_i \leftrightarrow r_j$) B then $\det(B) = -\det(A)$ ($i \neq$ j)
2. If A ~ ($r_i \leftarrow c r_i$) B then $\det(B) = c\det(A)$
3. If A ~ ($r_i \leftarrow r_i + r_j$) B then $\det(B) = \det(A)$ ($i \neq$ j)
4. Combining 3 with 2 we get

   If A ~ ($r_i \leftarrow r_i + c r_j$) B then $\det(B) = \det(A)$ ($i \neq$ j)

Proof. Omitted.

Remark. Thanks to the transposition law the theorem applies also to column rather than row operations.

Remark.
"A ~ ($r_i \leftarrow c r_i$) B" means "B has been obtained from A by replacing r_i of A with $c r_i$"

Theorem. (*det* versus not-quite-matrix-addition)

Suppose $s \in \{1,2,\ldots,n\}$ and A[i,j] = B[i,j] = C[i,j] for every i,j such that $j \neq s$ and C[i,s] = A[i,s] + B[i,s]. Then det(C) = det(A) + det (B).

Proof..

$$\det \begin{bmatrix} c_{1,1} & \cdots & a_{1,s} + b_{1,s} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & a_{2,s} + b_{2,s} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \cdots & a_{m,s} + b_{n,s} & \cdots & c_{n,n} \end{bmatrix} =$$

By Laplace expansion on column s

$$\sum_{i=1}^{n}(-1)^{i+s}(a_{i,s}+b_{i,s}) \det(C_{i,s}) = \sum_{i=1}^{n}(-1)^{i+s}a_{i,s} \det(C_{i,s}) + \sum_{i=1}^{n}(-1)^{i+s}b_{i,s} \det(C_{i,s}) = \det(A)+\det(B).$$

Warning. This is NOT about determinant of the sum of two matrices being equal to the sum of their determinants; **that is not true.** This is about determinant of a matrix whose ONE column is the sum of two vectors.

Theorem. (other properties of *det*)

For every $n \times n$ matrices A and B

1. det(A) $\neq$ 0 iff r(A) = n, in other words rows of A are linearly independent
2. If for every i,j such that $i > j$ $a_{i,j} = 0$ (all 0's below the main diagonal, triangular matrix) then det(A) = $a_{1,1} a_{2,2} \ldots a_{n,n}$
3. In particular, $det(I_{n,n}) = 1$
4. det(AB) = det(A) det(B)

Proof. Omitted.

Part 2 suggests a strategy for calculation of determinants of large matrices: row-reduce the matrix to a triangular form.

Determinant and systems of linear equations

Theorem. (Uniqueness theorem)

A system of *n* linear equations with *n* unknowns

$$(*)\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \ ... + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ ... + a_{2,n}x_n = b_2 \\ \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \ ... + a_{n,n}x_n = b_n \end{cases}$$

has a unique solution iff det(A) $\neq 0$

Proof.

It follows from the fact that the corresponding homogeneous system has unique solution Θ iff rank(A)=*n* which in turn is equivalent to *det*(A)$\neq 0$. Then, if (and that's a big IF) v_0 is a solution then all solutions *v* of $(*)$ look like $v = \Theta + v_0 = v_0$.

Warning. (NOT the usual one)

The uniqueness theorem is a "both ways" implication but is often misunderstood. The conclusion should be understood as "the set of solutions of $(*)$ has exactly one element". Hence the negation of this is (contrary to what many people believe) not

"if $\det(A) = 0$ then $(*)$ has no solutions"

but rather (remember de Morgan's Law!)

"if $\det(A) = 0$ then $(*)$ the set of solutions of $(*)$ does not have exactly one element"

which means either zero or more than one element. Look at this:

$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$ $\det\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ = 4−4 = 0 but the system has infinitely many solutions of the form $(t, 2 - t)$ where t is any real number.

Theorem. (Cramer's Rule)
Let A be an $n \times n$ matrix with det(A)$\neq$ 0 and let B be any $n \times 1$ matrix. Then the system of equations AX=B has unique solution $X = [x_1, x_2, \ldots , x_n]^T$ and for each i=1,2,…,n

$$x_i = \frac{detA_i}{detA},$$

where A_i is obtained by replacing i-th column of A with B.
Proof (skipped).

Example.

$$\begin{cases} 2x + 4y - z = 11 \\ -4x - 3y + 3z = -20 \\ 2x + 4y + 2z = 2 \end{cases} \quad |A| = \begin{vmatrix} 2 & 4 & -1 \\ -4 & -3 & 3 \\ 2 & 4 & 2 \end{vmatrix} = -12+16+24-6-24+32=30,$$

$$|A_1| = \begin{vmatrix} 11 & 4 & -1 \\ -20 & -3 & 3 \\ 2 & 4 & 2 \end{vmatrix} = -66+80+24-6-132+160=264-204=60,\ x = 2$$

$$|A_2| = \begin{vmatrix} 2 & 11 & -1 \\ -4 & -20 & 3 \\ 2 & 2 & 2 \end{vmatrix} = -80+8+66-40-12+88 = 162-132 = 30,\ y = 1$$

$$|A_3| = \begin{vmatrix} 2 & 4 & 11 \\ -4 & -3 & -20 \\ 2 & 4 & 2 \end{vmatrix} = -12-160-176+66+160+32 = 258-348 = -90,\ z = -3$$

Definition. (Inverse matrix)
Let A be an $n \times n$ matrix. If there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A$ = I then A^{-1} is called *the inverse* of A.

Fact. The inverse matrix for A, if it exists, is unique.

This follows from the very general fact the in every associative algebra the inverse element, if there is one, is unique.

(Remember? If p and q are both inverses for x and e is the identity then (px)q = eq = q and p(xq) = pe = p and since the operation is associative we have (px)q = p(xq) so p=q).

Theorem.
A matrix is A invertible iff $\det(A) \neq 0$.

Proof.($\Rightarrow$)
If A^{-1} exists then $\det(AA^{-1}) = \det(I) = 1 = \det(A)\det(A^{-1})$ hence both $\det(A)$ and $\det(A^{-1})$ are different from zero.

($\Leftarrow$)

If $\det(A) \neq 0$ then, from the uniqueness theorem, for every $n \times 1$ matrix B the system $AX = B$ has a (unique) solution.

$$A^{-1} = \mathrm{X} = \begin{bmatrix} \mathrm{x}_{1,1} & \mathrm{x}_{1,2} & \cdots & \mathrm{x}_{1,n} \\ \mathrm{x}_{2,1} & \mathrm{x}_{2,2} & \cdots & \mathrm{x}_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{x}_{n,1} & \mathrm{x}_{n,2} & \cdots & \mathrm{x}_{n,n} \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} & \cdots & \mathrm{a}_{1,n} \\ \mathrm{a}_{2,1} & \mathrm{a}_{2,2} & \cdots & \mathrm{a}_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{a}_{n,1} & \mathrm{a}_{n,2} & \cdots & \mathrm{a}_{n,n} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathrm{I}$$

Consider $\begin{bmatrix} \mathrm{a}_{1,1} & \mathrm{a}_{1,2} & \cdots & \mathrm{a}_{1,n} \\ \mathrm{a}_{2,1} & \mathrm{a}_{2,2} & \cdots & \mathrm{a}_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{a}_{n,1} & \mathrm{a}_{n,2} & \cdots & \mathrm{a}_{n,n} \end{bmatrix} \begin{bmatrix} \mathrm{x}_{1,1} \\ \mathrm{x}_{2,1} \\ \vdots \\ \mathrm{x}_{n,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. The system is uniquely solvable and the solution, X_1 is the first column of A^{-1}. The same can be said about the second, third and each next column of X and I. QED

The proof provides a method (two methods, really) for calculating A^{-1}(that's one reason I insist on doing proofs):

Method 1.

Row-reduce the following matrix to a row-canonical one

$$[A|I] = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & 1 & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & 0 & 0 & \dots & 1 \end{bmatrix} \sim \dots\sim\dots \sim \dots\sim$$

$$\sim\dots\sim \begin{bmatrix} 1 & 0 & \dots & 0 & x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ 0 & 1 & \dots & 0 & x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{bmatrix} = [I|A^{-1}]$$

This is always possible if A is invertible. So if A cannot be row-reduced to the identity matrix it proves that A is non-invertible.

Method 2.

Using Cramer's rule to calculate each $x_{i,j}$ of A^{-1}.

This method involves calculation of det(A) and n^2 determinants of the size $(n-1)\times(n-1)$. For large matrices it takes forever.

$x_{i,j}$ appears in *j*-th column of A^{-1} which means must consider

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_{1,j} \\ \vdots \\ x_{i,j} \\ \vdots \\ x_{n,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$= I_j$ where the solitary in I_j is in the *j*-th position. So, in order to find the *i*-th unknown we need divide the determinant of $A^*_{i,j}$ (A with *i*-th column replaced by I_j) by $\det(A)$.

$$\det A^{*}_{i,j} = \det \begin{bmatrix} a_{1,1} & \cdots & 0 & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & \textcircled{1} & \cdots & a_{j,n} \\ \vdots & & \vdots & \cdots & \vdots \\ a_{n,1} & & 0 & \cdots & a_{n,n} \end{bmatrix}$$

in j-th row and i-th column of $A^{*}_{i,j}$

If you do this determinant by *i*-th column, the only nonzero term in the Laplace expansion will be $(-1)^{i+j}$ times the determinant obtained by the removal of *j*-th row and *i*-th column from $A^{*}_{i,j}$. Here is the funny thing: A and $A^{*}_{i,j}$ only differ on the *i*-th column, which is being removed. Hence $\det A^{*}_{i,j} = (-1)^{i+j} \det A_{j,i}$ and, finally, $x_{i,j} = \frac{(-1)^{i+j} \det A_{j,i}}{\det A}$. In other words

$$A^{-1} = \frac{1}{\det A}\left[(-1)^{i+j} \det A_{i,j}\right]^{T}$$

Example.

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix}. \text{ Find } A^{-1}.$$

Method 1. (Gauss elimination)

$$\begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \boldsymbol{r_4 - r_2, -r_3 + 2r_2} \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 2 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \sim$$

$$\boldsymbol{r_1 - 2r_3, r_2 - r_3, r_4 - 2r_3} \begin{bmatrix} 0 & 0 & -7 & 1 & 1 & -4 & 2 & 0 \\ 1 & 0 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -6 & 1 & 0 & -5 & 2 & 1 \end{bmatrix} \boldsymbol{-r_1 + r_4}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 4 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -6 & 1 & 0 & -5 & 2 & 1 \end{bmatrix} \boldsymbol{r_2 + 2r_1, r_3 - 4r_1, r_4 + 6r_1}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -2 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 & 6 & -1 & -4 \\ 0 & 0 & 0 & 1 & -6 & -11 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 & 6 & -1 & -4 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -6 & -11 & 2 & 7 \end{bmatrix}$$

A^{-1}

Method 2. (Cramer's Rule, cofactors)

$\det A = 1$. We are cheating here, this is based on method 1. Only two transformations in the previous slide affected the determinant, in both cases they were like $-r_s + cr_t$ which really means two operations: *scale* r_s *by* (-1) and *add to the new* r_s *another row (perhaps scaled by some factor).* Scaling a row by (-1) changes the sign of the determinant and we did it twice.

Let's calculate just a single entry of A^{-1}, say $A^{-1}(2,3)$. According to the cofactor theorem $A^{-1}(2,3) = \frac{1}{detA}(-1)^{2+3}\det(A_{3,2})$

$$\det(A_{3,2}) = \begin{vmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{vmatrix} = 4 - 2 - 1 = 1, \text{ which}$$

means $A^{-1}[2,3]$ should be -1. We move back one slide and … surprise, surprise! it checks. And then you have to calculate the remaining 13 entries of A^{-1}